

# Simple epistemic planning: generalised gossiping

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**Abstract.** The gossip problem, in which information (known as secrets) must be shared among a certain number of agents using the minimum number of calls, is of interest in the conception of communication networks and protocols. We extend the gossip problem to arbitrary epistemic depths. For example, we may require not only that all agents know all secrets but also that all agents know that all agents know all secrets. We give optimal protocols for various versions of the generalised gossip problem, depending on the graph of communication links, in the case of two-way communications, one-way communications and parallel communication. We also study different variants which allow us to impose negative goals such as that certain agents must not know certain secrets. We show that in the presence of negative goals testing the existence of a successful protocol is NP-complete whereas this is always polynomial-time in the case of purely positive goals.

## 1 Introduction

We consider communication problems concerning  $n$  agents. We consider that initially, for  $i = 1, \dots, n$ , agent  $i$  has some information  $s_i$ , also known as this agent's secret since, initially, the other agents do not know this information. In many applications, this corresponds to information that agent  $i$  wishes to share with all other agents, such as agent  $i$ 's signature on a contract or the dates when agent  $i$  is available for a meeting. On the other hand, it may be confidential information which is only to be shared with a subset of the other agents, such as agent  $i$ 's telephone number, cryptographic key, password or credit card number. More mundanely, it could simply be some gossip that agent  $i$  wants to share. Indeed, the simplest version of the problem in which all agents want to communicate their secrets to all other agents (using the minimum number of communications) is traditionally known as the gossip problem. Several variants have been studied in the literature, and a survey of these alternatives and the associated results has been published [14].

The gossip problem is a particular case of a multiagent epistemic planning problem. We view it as an epistemic counterpart of blockworld problems where the complexity of epistemic planning problems can be illustrated in a nice way. We demonstrate this by studying several variants of the classic problem: by supposing that not all pairs of agents can communicate directly, by allowing parallel or one-way communications, and by introducing the notion of confidential information which should not be shared with all other agents. However, our main contribution is to study the gossip problem at different epistemic depths. In the classic gossip problem, the goal is for all agents to know all secrets (which corresponds to epistemic depth 1). The equivalent goal at epistemic depth 2 is that all agents know

that all agents know all the secrets; at depth 3, all agents must know that all agents know that all agents know all the secrets. For example, in a commercial setting, if the secrets are the agents' agreement to the terms of a joint contract, then an agent may not authorize expenditure on the project before knowing that all other agents know that all agents agree to the terms of the contract. We provide algorithms for these variants and establish their optimality in most of the cases.

The paper is organized as follows. In the next section we formally introduce epistemic planning and the epistemic version of the classic gossip problem  $\text{Gossip}_G(d)$ . In Section 3 we study the properties of  $\text{Gossip}_G(d)$ . In Section 4 we turn our attention to the version of this problem in which all communications are one-way (such as e-mails rather than telephone calls). In Section 5 we study a parallel version in which calls between different agents can take place simultaneously. In each of these three cases, we give a protocol which is optimal (given certain conditions on the graph  $G$ ) assuming we want to attain all positive epistemic goals up to depth  $d$ . We then consider versions of the gossip problem with some negative goals. This version of the gossip problem has obvious applications concerning confidential information which is only to be broadcast to a subset of the other agents. In Section 6 we show that determining the existence of a plan which attains a mixture of positive and negative goals is NP-complete. In Section 7 we show that allowing agents to change their secrets (when secrets correspond, for example, to passwords or telephone numbers) allows more problems to be solved, but testing the existence of a plan remains NP-complete. We conclude with a discussion in Section 8.

## 2 Epistemic planning and the gossip problem

Dynamic Epistemic Logic DEL [24] provides a formal framework for the representation of knowledge and update of knowledge, and several recent approaches to multi-agent planning are based on it, starting with [5, 21]. While DEL provides a very expressive framework, it was unfortunately proven to be undecidable even for rather simple fragments of the language [2, 7]. Some decidable fragments were studied, most of which focused on public events [21, 26]. However, the gossip problem requires private communication. We here consider a simple fragment of the language of DEL where the knowledge operator can only be applied to literals. Similar approaches to epistemic planning can be found in [17, 22].

Here we propose a more direct model. We use the notation  $K_i s_j$  to represent the fact that agent  $i$  knows the secret of  $j$ , the notation  $K_i K_j s_k$  to represent the fact that agent  $i$  knows that agent  $j$  knows the secret of  $k$ , etc. We use the term *positive fluent* for any epistemic proposition of the form  $K_{i_1} \dots K_{i_r} s_j$ . If we consider the secrets  $s_i$  as constants and that agents never forget, then positive fluents, once true, can never become false. A negative fluent  $\neg(K_{i_1} \dots K_{i_r} s_j)$

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can, of course, become false. Note that these fluents are not modal formulas of epistemic logic;  $K_i$  is not a modal operator. They should simply be viewed as independent propositional variables.

A *planning problem* consists of an initial state (a set of fluents  $I$ ), a set of actions and a set of goals (another set of fluents  $Goal$ ). Each action has a (possibly empty) set of preconditions (fluents that must be true before the action can be executed) and a set of effects (positive or negative fluents that will be true after the execution of the action). A *solution plan* (or protocol) is a sequence of actions which when applied in this order to the initial state  $I$  produces a state in which all goals in  $Goal$  are true. We use the term *epistemic planning problem* when we need to emphasize that fluents may include the operators  $K_i$  ( $i = 1, \dots, n$ ). A simple epistemic goal is that all agents know all the secrets, i.e.  $\forall i, j \in \{1, \dots, n\}, K_i s_j$ . A higher-level epistemic goal is  $\forall i, j, k \in \{1, \dots, n\}, K_i K_j s_k$ , i.e. that all agents know that all agents know all the secrets.

The *gossip problem* on  $n$  agents and a graph  $G = \langle \{1, \dots, n\}, E_G \rangle$  is the epistemic planning problem in which the actions are  $CALL_{i,j}$  for  $\{i, j\} \in E_G$  (i.e. there is an edge between  $i$  and  $j$  in  $G$  if and only if they can call each other) and the initial state contains  $K_i s_i$  for  $i = 1, \dots, n$  (and implicitly all fluents of the form  $K_{i_1} \dots K_{i_r} s_j$  with  $i_r = j$ , together with all fluents of the form  $\neg(K_{i_1} \dots K_{i_r} s_j)$  with  $i_r \neq j$ ). The action  $CALL_{i,j}$  has no preconditions and its effect is that agents  $i$  and  $j$  share all their knowledge. We go further and assume that the two agents know that they have shared all their knowledge, so that, if we had  $K_i f$  or  $K_j f$  before the execution of  $CALL_{i,j}$ , for any fluent  $f$ , then we have  $K_{i_1} \dots K_{i_r} f$  just afterwards, for any  $r$  and for any sequence  $i_1, \dots, i_r \in \{i, j\}$ . The assumption that two agents share all their knowledge when they communicate may appear unrealistic, but, if the aim is to broadcast information using the minimum number of calls, this is clearly the best strategy. Furthermore, in applications in which some secrets must not be divulged to all agents, it is important to study the worst-case scenario in which all information is exchanged whenever a communication occurs.

Let  $Gossip\text{-}pos_G(d)$  be the gossip problem on a graph  $G$  in which the goal is a conjunction of positive fluents of the form  $(K_{i_1} \dots K_{i_r} s_j)$  ( $1 \leq r \leq d$ ). Thus, the parameter  $d$  specifies the maximum epistemic depth of goals. We use  $Gossip_G(d)$  to denote the specific problem in which *all* such goals must be attained. For any fixed  $d \geq 1$ ,  $Gossip\text{-}pos_G(d)$  can be solved in polynomial time since it can be coded as a classic STRIPS planning problem in which actions have no preconditions, and all effects of actions and all goals are positive [6]. Indeed, a necessary and sufficient condition for a solution plan to exist is that, for all goals  $(K_{i_1} \dots K_{i_r} s_j)$ , there is a path in  $G$  from  $j$  to  $i_1$  passing through  $i_r, \dots, i_2$  (in this order). Let  $Gossip\text{-}neg_G(d)$  be the gossip problem in which the goal is a conjunction of goals of the form  $(K_{i_1} \dots K_{i_r} s_j)$  ( $1 \leq r \leq d$ ) or  $\neg(K_{i_1} \dots K_{i_r} s_j)$  ( $1 \leq r \leq d$ ). We write  $Gossip\text{-}pos(d)$ ,  $Gossip(d)$  and  $Gossip\text{-}neg(d)$  to denote the corresponding problems in which the graph  $G$  is part of the input. Versions with one-way and parallel communication will be defined in sections 4 and 5. In Section 7 we will define a version where secrets can change truth value.

### 3 Minimising the number of calls for positive goals

In this section we consider the gossip problem with only positive goals. The minimal number of calls to obtain the solution of  $Gossip_G(1)$  is either  $2n - 4$  if the graph  $G$  contains a quadrilateral (a cycle of length 4) as a subgraph, or  $2n - 3$  in the general case [13]. We first give a simple protocol for any connected graph before giving

protocols requiring many less calls for special cases of  $G$ .

**Proposition 1** *If the graph  $G$  is connected, then for  $n \geq 2$  and  $d \geq 1$ , any instance of  $Gossip\text{-}pos_G(d)$  has a solution of length no greater than  $d(2n - 3)$  calls.*

**Proof:** Since  $G$  is connected, it has a spanning tree  $\mathcal{T}$ . Let the root of  $\mathcal{T}$  be 1. Since  $n \geq 2$ , there is a node 2 that is connected to 1. Let  $\mathcal{T}_2$  be the subtree rooted in 2 and let  $\mathcal{T}_1$  be the rest of  $\mathcal{T}$ , i.e., 1 together with all its subtrees except  $\mathcal{T}_2$ . Let  $|\mathcal{T}_i|$  be the number of edges in tree  $\mathcal{T}_i$ .

Consider the following protocol, composed of a total of  $2d$  passes. Each pass either consists in calls that go upwards in  $\mathcal{T}$  followed by a call between 1 and 2, or consists in calls that go downward.

odd passes:  $|\mathcal{T}_1|$  calls upwards in  $\mathcal{T}_1$ , starting with the leaves;  
 $|\mathcal{T}_2|$  calls upwards in  $\mathcal{T}_2$ , starting with the leaves;  
 $CALL_{1,2}$   
 even passes:  $|\mathcal{T}_1|$  calls downwards in  $\mathcal{T}_1$ , starting with 1;  
 $|\mathcal{T}_2|$  calls downwards in  $\mathcal{T}_2$ , starting with 2

After  $k$  passes:

- if  $k = 2m - 1$  then  $K_1 K_{i_1} \dots K_{i_{m-1}} s_j$  and  $K_2 K_{i_1} \dots K_{i_{m-1}} s_j$  are true for all  $i_1, \dots, i_{m-1}, j$ ;
- if  $k = 2m$  then  $K_{i_1} \dots K_{i_m} s_j$  is true for all  $i_1, \dots, i_m, j$ ;

So the goal is attained after  $2d$  passes. Since the odd passes have  $|\mathcal{T}_1| + |\mathcal{T}_2| + 1 = |\mathcal{T}| = n - 1$  calls and the even passes have  $n - 2$  calls, this gives us a total of  $d(2n - 3)$  calls. ■

In fact, for  $d \geq 2$ , we require considerably less than  $d(2n - 3)$  calls if  $G$  has a Hamiltonian path.

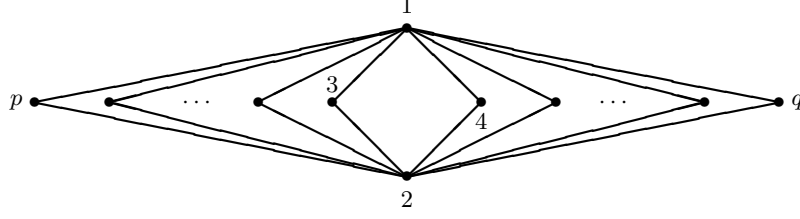
**Proposition 2** *If the graph  $G$  has a Hamiltonian path, then any instance of  $Gossip\text{-}pos_G(d)$  has a solution of length no greater than  $1 + (d + 1)(n - 2)$ .*

**Proof:** Let  $\pi$  be the Hamiltonian path in  $G$ . Number the vertices of  $G$  from 1 to  $n$  in the order they are visited in  $\pi$ .

Consider the following protocol:

first pass:  $CALL_{i,i+1}$  (for  $i = n - 1, \dots, 1$ ),  
 then  $CALL_{i,i+1}$  (for  $i = 2, \dots, n - 1$ )  
 second pass:  $CALL_{i,i+1}$  (for  $i = n - 2, \dots, 1$ )  
 third pass:  $CALL_{i,i+1}$  (for  $i = 2, \dots, n - 1$ )  
 $\vdots$   
 even passes:  $CALL_{i,i+1}$  (for  $i = n - 2, \dots, 1$ )  
 odd passes:  $CALL_{i,i+1}$  (for  $i = 2, \dots, n - 1$ )  
 $\vdots$

It is not difficult to see that the first pass establishes  $K_i s_j$  for all  $i, j$ , and indeed it establishes both  $K_{n-1} K_i s_j$  and  $K_n K_i s_j$  for all  $i, j$  since  $CALL_{n-1,n}$  is the last communication in this pass. By a straightforward induction argument, we can show that the  $m$ th pass, for  $m$  even, establishes  $K_{i_1} \dots K_{i_m} s_j$  for all  $i_1, \dots, i_m, j$ , and indeed that the  $m$ th pass establishes both  $K_1 K_{i_1} \dots K_{i_m} s_j$  and  $K_2 K_{i_1} \dots K_{i_m} s_j$  for all  $i_1, \dots, i_m, j$  since  $CALL_{1,2}$  is the last communication in this pass. Similarly, when  $m$  is odd, the  $m$ th pass establishes  $K_{i_1} \dots K_{i_m} s_j$  for all  $i_1, \dots, i_m, j$ , and indeed both  $K_{n-1} K_{i_1} \dots K_{i_m} s_j$  and  $K_n K_{i_1} \dots K_{i_m} s_j$  for all  $i_1, \dots, i_m, j$ . The above plan then establishes, after  $d$  passes, all possible depth- $d$



**Figure 1.** A complete bipartite graph  $K_{2,n-2}$ .

epistemic goals. The number of CALL actions in this plan is  $2n - 3$  in the first pass and  $n - 2$  in each subsequent pass, which makes  $2n - 3 + (d - 1)(n - 2) = 1 + (d + 1)(n - 2)$  in total after  $d$  passes. ■

The first pass of the protocol given in the proof of Proposition 2 scans agents from  $n$  to 1 and then from 1 to  $n$ , whereas each subsequent pass consists of a single scan. We can explain this by the fact that the purpose of the first scan is to group secrets together; the purpose of the second scan is then to broadcast this grouped information to all agents. What is surprising is that we only require one scan for each subsequent increment in the epistemic depth  $d$ .

Determining the existence of a Hamiltonian path is known to be NP-complete [11]. However, this does not necessarily imply that finding an optimal solution for Gossip-pos( $d$ ) (the problem in which the graph  $G$  is part of the input) is NP-hard. One reason is that we do not actually need a Hamiltonian path to obtain a plan of length  $1 + (d + 1)(n - 2)$ . In fact, in the protocol given in the proof of Proposition 2 we can replace the actions  $\text{CALL}_{i,i+1}$  ( $i = 1, \dots, n - 1$ ) by any sequence  $\text{CALL}_{j_i,i+1}$  ( $i = 1, \dots, n - 1$ ) such that  $j_1 = 1$  and  $\forall i = 1, \dots, n - 2, \{j_i, i + 1\} \cap \{j_{i+1}, i + 2\} \neq \emptyset$ . Another reason why the existence of a Hamiltonian path is not necessarily critical is that we can often actually do better. Indeed, the value  $1 + (d + 1)(n - 2)$  is not necessarily optimal, since for certain graphs we can achieve  $(d + 1)(n - 2)$ , i.e. one call less.

The graph shown in Figure 1 is the complete bipartite graph with parts  $\{1, 2\}$ ,  $\{3, \dots, n\}$ , and is denoted in graph theory by  $K_{2,n-2}$ . We now show that there is a protocol which achieves  $(d + 1)(n - 2)$  calls provided  $G$  contains  $K_{2,n-2}$  as a subgraph. This subsumes a previous result which was given only for the case of a complete graph  $G$  [15].

**Proposition 3** *For  $n \geq 4$ , if the  $n$ -vertex graph  $G$  has  $K_{2,n-2}$  as a subgraph, then any instance of Gossip-pos $_G(d)$  has a solution of length no greater than  $(d + 1)(n - 2)$ .*

**Proof:** Suppose that the two parts of  $K_{2,n-2}$  are  $\{1, 2\}$ ,  $\{3, \dots, n\}$ . We choose an arbitrary partition of the vertices  $3, \dots, n$  into two non-empty sets  $L, R$ . We can number the vertices so that  $\min(L) = 3$  and  $\min(R) = 4$ . Denote  $\max(L)$  by  $p$  and  $\max(R)$  by  $q$  (as shown in Figure 1).

Consider the following protocol:

odd passes:       $\text{CALL}_{1,3} \dots \text{CALL}_{1,p} \quad \text{CALL}_{2,4} \dots \text{CALL}_{2,q}$   
 even passes:     $\text{CALL}_{1,q} \dots \text{CALL}_{1,4} \quad \text{CALL}_{2,p} \dots \text{CALL}_{2,3}$

In other words: the odd passes are composed of  $\text{CALL}_{1,x}$  for each  $x \in L$  in increasing order of  $x$ , followed by  $\text{CALL}_{2,y}$  for each  $y \in R$  in increasing order of  $y$ ; and the even passes are composed of  $\text{CALL}_{1,y}$  for each  $y \in R$  in decreasing order of  $y$ , followed by  $\text{CALL}_{2,x}$  for each  $x \in L$  in decreasing order of  $x$ . The length of this plan after  $d + 1$  passes is  $(d + 1)(|L| + |R|) = (d + 1)(n - 2)$ . It

therefore only remains to show that  $(d + 1)$  passes are sufficient to establish all possible depth- $d$  epistemic goals. A positive epistemic fluent of the form  $K_{i_1} \dots K_{i_d} s_j$ , for agents  $i_1, \dots, i_d, j$ , has depth  $d$ . In particular,  $s_j$  has depth 0.

For  $m \geq 1$ , let  $H_m$  be the hypothesis that after  $m$  passes, for all depth  $m - 1$  positive epistemic fluents  $f$ , we have

$$\begin{aligned} (K_1 f \vee K_q f) \quad \wedge \quad (K_2 f \vee K_p f) & \quad \text{if } m \text{ is odd} \\ (K_1 f \vee K_3 f) \quad \wedge \quad (K_2 f \vee K_4 f) & \quad \text{if } m \text{ is even} \end{aligned}$$

It is not difficult to see that  $H_1$  is true after the first pass. For  $H_m \Rightarrow H_{m+1}$ , suppose  $m$  is even. By  $H_m$ , after pass  $m$ , we have  $K_1 f \vee K_3 f$  for all positive fluents  $f$  of depth  $m - 1$ . Thus the first call of pass  $m + 1$ ,  $\text{CALL}_{1,3}$ , makes 1 and 3 know all fluents of depth  $m - 1$ . After  $\text{CALL}_{1,p}$ , 1 and  $p$  know that 1, 3,  $\dots$ ,  $p$  know all fluents of depth  $m - 1$ . The same goes for 2: since we have  $K_2 f \vee K_4 f$  by  $H_m$ , after  $\text{CALL}_{2,4}$ , 2 and 4 know all fluents of depth  $m - 1$ . At the end of pass  $m + 1$  (after  $\text{CALL}_{2,q}$ ), 2 and  $q$  know that 2, 4,  $\dots$ ,  $q$  know all fluents of depth  $m - 1$ . Thus for any fluent  $f$  of depth  $m$ , either 1 knows  $f$  or  $q$  knows  $f$ , and either 2 knows  $f$  or  $p$  knows  $f$ , that is,  $H_{m+1}$ . The reasoning is similar for  $m$  odd. The above plan therefore establishes, after  $d + 1$  passes, all possible depth- $d$  epistemic goals. ■

Observe that the complete graph on  $n \geq 4$  vertices has  $K_{2,n-2}$  as a subgraph. Furthermore, detecting whether an arbitrary graph  $G$  has  $K_{2,n-2}$  as a subgraph can clearly be achieved in polynomial time, since it suffices to test for each pair of vertices  $\{i, j\}$  whether or not  $G$  contains all edges of the form  $\{u, v\}$  ( $u \in \{i, j\}$ ,  $v \in \{1, \dots, n\} \setminus \{i, j\}$ ).

Recall that Gossip $_G(d)$  denotes the version of Gossip-pos $_G(d)$  in which the goal consists of all depth- $d$  positive epistemic fluents. We can, in fact, show that the solution plan given in the proof of Proposition 3 is optimal for Gossip $_G(d)$ .

**Theorem 4** *The number of calls required to solve Gossip $_G(d)$  (for any graph  $G$ ) is at least  $(d + 1)(n - 2)$ .*

**Proof:** Consider any solution plan for Gossip $_G(d)$ . The goal of Gossip $_G(d)$  is to establish  $T_{d+1}$  (where  $T_r$  is the conjunction of  $K_{i_1} \dots K_{i_{r-1}} s_{i_r}$  for all  $i_1, \dots, i_r \in \{1, \dots, n\}$ ).

We give a proof by induction. Suppose that at least  $(r + 1)(n - 2)$  calls are required to establish  $T_{r+1}$ . This is true for  $r = 1$  because it takes at least a sequence of  $2n - 4$  calls to establish  $T_2$  (each agent knows the secret of each other agent) [3, 12, 23]. For general  $r$  and without loss of generality, suppose that before the last call to establish it,  $T_{r+1}$  was false because of lack of knowledge of agent  $j$  (i.e.  $K_j T_r$  was false). By induction hypothesis this is at least the  $((r + 1)(n - 2) - 1)$ -th call. This call involves  $j$  and another agent, say  $i$ , and establishes not only  $T_{r+1}$ , but also  $K_j T_{r+1}$  and  $K_i T_{r+1}$ . However,  $\neg K_k T_{r+1}$  holds both before and after this call,

for the agents  $k$  distinct from  $i$  and  $j$ . To establish  $T_{r+2}$ , it is necessary to distribute  $T_{r+1}$  from  $i$  and  $j$  to other agents and this takes at least  $n - 2$  calls. Hence, at least  $(r + 2)(n - 2)$  calls are required in total to establish  $T_{r+2}$ . By induction on  $r$ , it takes at least a sequence of  $(d+1)(n-2)$  calls to establish  $T_{d+1}$ . ■

## 4 One-way communications

We can consider a different version of the gossip problem, which we denote by Directional-gossip, in which communications are one-way. Whereas a telephone call is essentially a two-way communication, e-mails and letters are essentially one-way. We now consider the case in which the result of  $\text{CALL}_{i,j}$  is that agent  $i$  shares all his knowledge with agent  $j$  but agent  $i$  receives no information from agent  $j$ . Indeed, to be consistent with communication by e-mail, in which the sender cannot be certain when (or even if) an e-mail will be read by the receiver, we assume that after  $\text{CALL}_{i,j}$ , agent  $i$  does not even gain the knowledge that agent  $j$  knows the information that agent  $i$  has just sent in this call.

Clearly, Directional-gossip- $\text{pos}_G(d)$  can be solved in polynomial time, since any solution plan for Gossip- $\text{pos}_G(d)$  can be converted into a solution plan for Directional-gossip- $\text{pos}_G(d)$  by replacing each two-way call by two one-way calls. What is surprising is that the exact minimum number of calls to solve Directional-gossip- $\text{pos}_G(d)$  is often much smaller than this and indeed often very close to the minimum number of calls required to solve Gossip- $\text{pos}_G(d)$ . We consider, in particular, the hardest version of Directional-gossip- $\text{pos}_G(d)$ , in which the aim is to establish all epistemic goals of depth  $d$ . Let  $T_r$  be the conjunction of  $K_{i_1} \dots K_{i_{r-1}} s_{i_r}$  for all  $i_1, \dots, i_r \in \{1, \dots, n\}$ , and let Directional-gossip- $\text{pos}_G(d)$  denote the directional gossip problem whose goal is to establish  $T_{d+1}$ .

In the directional version, the graph of possible communications is now a directed graph  $G$ . Let  $\overline{G}$  be the graph with the same  $n$  vertices as the directed graph  $G$  but with an edge between  $i$  and  $j$  if and only if  $G$  contains the two directed edges  $(i, j)$  and  $(j, i)$ . It is known that if the directed graph  $G$  is strongly connected, the minimal number of calls for Directional-gossip- $\text{pos}_G(1)$  is  $2n - 2$  [13]. We now generalise this to arbitrary  $d$  under an assumption about the graph  $\overline{G}$ .

**Proposition 5** *For all  $d \geq 1$ , if  $\overline{G}$  contains a Hamiltonian path, then any instance of Directional-gossip- $\text{pos}_G(d)$  has a solution of length no greater than  $(d + 1)(n - 1)$ .*

**Proof:** We give a protocol which establishes all positive goals of epistemic depth up to  $d$ . Without loss of generality, suppose that the Hamiltonian path in  $\overline{G}$  is  $1, 2, \dots, n$ . Consider the plan consisting of  $d + 1$  passes according to the following protocol:

odd passes       $\text{CALL}_{i,i+1}$  (for  $i = 1, \dots, n - 1$ )  
even passes       $\text{CALL}_{i+1,i}$  (for  $i = n - 1, \dots, 1$ )

We show by a simple inductive proof that this protocol is correct for any  $d \geq 1$ . Recall that  $T_r$  is the conjunction of  $K_{i_1} \dots K_{i_{r-1}} s_{i_r}$  for all  $i_1, \dots, i_r \in \{1, \dots, n\}$ . Consider the hypothesis  $H(r)$ : at the end of pass  $r$ , if  $r$  is odd we have  $K_n T_r$  and if  $r$  is even we have  $K_1 T_r$ . Clearly,  $H(1)$  is true since at the end of the first pass agent  $n$  knows all the secrets  $s_i$  ( $i = 1, \dots, n$ ). If  $r$  is odd and  $H(r)$  holds, then at the end of pass  $r + 1$ , all agents know  $T_r$  and furthermore agent 1 knows this (i.e.  $K_1 T_{r+1}$ ). A similar argument shows that  $H(r) \Rightarrow H(r + 1)$  when  $r$  is even. By induction,  $H(r)$  holds for all  $r = 1, \dots, d + 1$ . For  $K_n T_r$  or  $K_1 T_r$  to hold, we must have  $T_r$ .

Thus after  $d + 1$  passes, and  $(d + 1)(n - 1)$  calls, we have the goal  $T_{d+1}$ . ■

However, as pointed out in Section 3, determining the existence of a Hamiltonian path in a graph is NP-complete [11].

We now show that the solution plan given in the proof of Proposition 5 is optimal even for a complete digraph  $G$ .

**Theorem 6** *The number of calls required to solve Directional-gossip- $\text{pos}_G(d)$  (for any digraph  $G$ ) is at least  $(d + 1)(n - 1)$ .*

**Proof:** Consider any solution plan for Directional-gossip- $\text{pos}_G(d)$ . The goal of Directional-gossip- $\text{pos}_G(d)$  is to establish  $T_{d+1}$  (the conjunction of  $K_{i_1} \dots K_{i_d} s_{i_{d+1}}$  for all  $i_1, \dots, i_{d+1} \in \{1, \dots, n\}$ ). Consider the following claims (for  $1 \leq r \leq d$ ):

- C1( $r$ )** after  $r(n - 1) - 1$  calls no agent knows  $T_r$ .
- C2( $r$ )** after  $r(n - 1)$  calls at most one agent knows  $T_r$ .
- C3( $r$ )** at least  $(r + 1)(n - 1)$  calls are required to establish  $T_{r+1}$ .

C1(1) is true because  $T_1$  is the conjunction of all the secrets  $s_j$  and no agent  $i$  can know all the secrets after only  $n - 2$  calls since after  $n - 2$  calls, there is necessarily some agent  $i' \neq i$  who has not communicated his secret to anyone. Let  $r \in \{1, \dots, d\}$ . We will show  $\text{C1}(r) \Rightarrow \text{C2}(r) \Rightarrow \text{C3}(r) \Rightarrow \text{C1}(r + 1)$ .

**C1( $r$ )  $\Rightarrow$  C2( $r$ ):** Straightforward, since during one call only one agent gains knowledge.

**C2( $r$ )  $\Rightarrow$  C3( $r$ ):** Suppose that C2( $r$ ) holds, i.e. after  $r(n - 1)$  calls at most one agent knows  $T_r$ . This means that the other  $n - 1$  agents require some information in order to know  $T_r$ . Hence we require at least  $n - 1$  other calls, i.e.  $(r + 1)(n - 1)$  calls in total, to establish  $T_{r+1}$ .

**C3( $r$ )  $\Rightarrow$  C1( $r + 1$ ):** Suppose C3( $r$ ) is true and C1( $r + 1$ ) is false. Then we require at least  $(r + 1)(n - 1)$  calls to establish  $T_{r+1}$  but after  $(r + 1)(n - 1) - 1$  calls some agent  $i$  knows  $T_{r+1}$ . There is clearly a contradiction since agent  $i$  cannot know something which is false.

This completes the proof by induction that at least  $(d + 1)(n - 1)$  calls are required to establish  $T_{d+1}$ , since this corresponds exactly to C3( $d$ ). ■

It is worth pointing out that, by Theorem 4, the optimal number of 2-way calls is only  $d + 1$  less than the optimal number of one-way calls and is hence independent of  $n$ , the number of agents.

## 5 Parallel communications

An interesting variant, which we call Parallel-gossip- $\text{pos}_G(d)$ , is to consider time steps instead of calls, and thus suppose that in each time step several calls are executed in parallel. However, each agent can only make one call in any given time step. We denote by Parallel-gossip- $\text{pos}_G(d)$  the problem of establishing all depth- $d$  positive epistemic fluents. For Parallel-gossip- $\text{pos}_G(1)$  on a complete graph  $G$ , if the number of agents  $n$  is even, the time taken (in number of steps) is  $\lceil \log_2 n \rceil$ , and if  $n$  is odd, it is  $\lceil \log_2 n \rceil + 1$  [4, 20, 16]. We now generalise this to the case of arbitrary epistemic depth  $d$ .

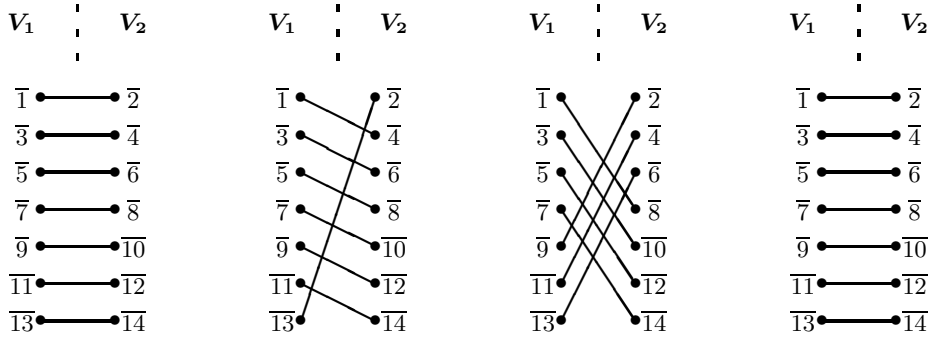


Figure 2. The four steps in the first pass of the parallel protocol for  $n = 14$ .

**Proposition 7** For  $n \geq 2$ , if the  $n$ -vertex graph  $G$  has the complete bipartite graph  $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$  as a subgraph, then any instance of Parallel-gossip-pos $_G(d)$  has a solution with  $d(\lceil \log_2 n \rceil - 1) + 1$  time steps if  $n$  is even, or  $d\lceil \log_2 n \rceil + 1$  time steps if  $n$  is odd.

**Proof:** Suppose that  $G$  has  $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$  as a subgraph. So we can partition the vertex set of  $G$  into two subsets  $V_1$  and  $V_2$  of size  $\lceil n/2 \rceil$  and  $\lfloor n/2 \rfloor$ , respectively, such that  $G$  has an edge  $\{i, j\}$  for each  $i \in V_1$  and  $j \in V_2$ . We can number agents by elements of the ring  $\mathbb{Z}/n\mathbb{Z} = \{\bar{1}, \dots, \bar{n}\}$  so that for all  $i \in \mathbb{Z}$ ,  $\bar{2i+1} \in V_1$  and  $\bar{2i+2} \in V_2$ , where  $\bar{x}$  denotes the corresponding element of  $\mathbb{Z}/n\mathbb{Z}$  for all  $x \in \mathbb{Z}$ . We consider separately the cases  $n$  even and  $n$  odd.

For even  $n$ , consider the following protocol:

first pass:

For each step  $s$  from 1 to  $\lceil \log_2 n \rceil$ :  
 $\forall i \in \{0, \dots, (\frac{n}{2} - 1)\}$ , CALL  $\overline{2i+1, 2i+2^s}$

subsequent passes:

Reorder even agents according to the permutation  $\pi$   
 given by  $\pi(2i + 2^{\lceil \log_2 n \rceil}) = \bar{2i+2}$ ;

Proceed as in the first pass but only for steps  $s$  from 2 to  $\lceil \log_2 n \rceil$

The first pass of this protocol is illustrated in Figure 2 for  $n = 14$ . Calls are represented by a line joining two agents.

In the first pass, because of the calls CALL  $\overline{2i+1, 2i+2}$ , the first step establishes for all  $i \in \mathbb{Z}$ ,  $K_{\overline{2i+1} s_{2i+2}}$  and  $K_{\overline{2i+2} s_{2i+1}}$ . Suppose that after step  $s$ , for all  $i \in \mathbb{Z}$ , we have the conjunction of  $K_{\overline{2i+1} s_j}$  and  $K_{\overline{2i+2^s} s_j}$  for all  $j \in \{\overline{2i+1}, \dots, \overline{2i+2^s}\}$ . We have just seen that this is true for  $s = 1$  (given that each agent knows his own secret). In particular, if we replace  $i$  by  $i + 2^{s-1}$  we have  $K_{\overline{2i+2^{s-1}+1} s_j}$  and  $K_{\overline{2i+2^{s-1}+2} s_j}$  for all  $j \in \{\overline{2i+2^{s-1}+1}, \dots, \overline{2i+2^{s-1}+2}\}$ . At step  $s + 1$ , we make the calls CALL  $\overline{2i+1, 2i+2^{s+1}}$  for all  $i \in \mathbb{Z}$ , and this establishes  $K_{\overline{2i+1} s_j}$  and  $K_{\overline{2i+2^{s+1}} s_j}$  for all  $j \in \{\overline{2i+1}, \dots, \overline{2i+2^{s+1}}\}$ . By induction on  $s$ , it is easily seen that after  $\lceil \log_2 n \rceil$  steps, for all  $i \in \mathbb{Z}$ , we have  $K_{\overline{2i+1} s_j}$  and  $K_{\overline{2i+2} s_j}$  for all  $j \in \mathbb{Z}/n\mathbb{Z}$ . This means that at the end of the first pass  $\forall i, j \in \{\overline{2i+1}, \dots, \overline{2i+2^{s+1}}\}$ ,  $K_i s_j$ .

Let  $T_r$  be the conjunction of  $K_{j_1} \dots K_{j_{r-1} s_{j_r}}$  for all  $j_1, \dots, j_r \in \mathbb{Z}/n\mathbb{Z}$ . We have just seen that after the first pass  $T_2$  is true. Suppose that at the end of pass  $r$ ,  $T_{r+1}$  is true. For the next pass  $r + 1$ , CALL  $\overline{2i+1, 2i+2^{\lceil \log_2 n \rceil}}$  are the calls in last step of the previous pass  $r$ . Hence, after reordering even agents so that  $\overline{2i+2^{\lceil \log_2 n \rceil}}$  replaces  $\overline{2i+2}$ , we already have for all  $i \in \mathbb{Z}$ ,  $K_{\overline{2i+1} s_{\overline{2i+2}}}$  and

$K_{\overline{2i+2} s_{\overline{2i+1}}} T_r$ . We then proceed as for the first pass replacing  $s_j$  by  $K_j T_r$  to establish  $T_{r+2}$  in  $\lceil \log_2 n \rceil - 1$  more steps.

It therefore takes  $d$  passes to establish all possible depth- $d$  epistemic goals  $T_{d+1}$ . The first pass takes  $\lceil \log_2 n \rceil$  steps and the next  $d - 1$  passes  $\lceil \log_2 n \rceil - 1$  steps, making a total of  $d(\lceil \log_2 n \rceil - 1) + 1$  steps.

For odd  $n$ , one can place the first  $2^{\lceil \log_2 n \rceil}$  agents in a subset  $V_{\text{first}}$ , the others being in a subset  $V_{\text{last}}$  (see the example in Figure 3 for  $n = 13$ ). Consider the following protocol:

preliminary step:

Each agent in  $V_1 \cap V_{\text{last}}$  calls one agent in  $V_2 \cap V_{\text{first}}$ ,  
 and each agent in  $V_2 \cap V_{\text{last}}$  calls one agent in  $V_1 \cap V_{\text{first}}$

subsequent passes:

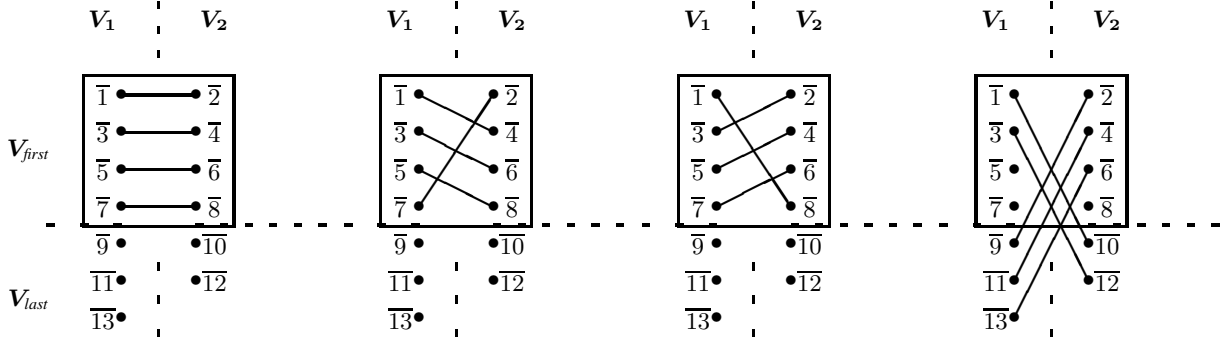
Proceed in  $V_{\text{first}}$  as for the first pass of even case in  $\mathbb{Z}/2^{\lceil \log_2 n \rceil} \mathbb{Z}$ ;  
 Each agent in  $V_1 \cap V_{\text{last}}$  calls one agent in  $V_2 \cap V_{\text{first}}$ ,  
 and each agent in  $V_2 \cap V_{\text{last}}$  calls one agent in  $V_1 \cap V_{\text{first}}$

A typical pass of this protocol is illustrated in Figure 3. The preliminary step is the step on the right of this figure.

In the preliminary step, all agents  $i_1, \dots, i_m \in V_{\text{last}}$  distribute their knowledge to some agents  $j_1, \dots, j_m \in V_{\text{first}}$ . Hence, after this step we have  $K_{j_k} s_{i_k}$  for all  $k \in \{1, \dots, m\}$ . For each subsequent pass  $r$ , it takes  $\lceil \log_2 n \rceil = \lceil \log_2 n \rceil - 1$  steps to distribute knowledge from all agents in  $V_{\text{first}}$  (hence, in  $V_{\text{last}}$  too because of the previous step) and establish  $K_j T_r$  for all  $j \in V_{\text{first}}$ . Then agents  $j_1, \dots, j_m \in V_{\text{first}}$  respectively call the agents  $i_1, \dots, i_m \in V_{\text{last}}$  in one more step to establish  $T_{r+1}$ . These last calls also establish  $K_{j_k} K_{i_k} T_r$  for all  $k \in \{1, \dots, m\}$  if necessary for the next pass  $r + 1$ .

It takes one preliminary step and  $d$  passes of  $\lceil \log_2 n \rceil$  steps to establish all possible depth- $d$  epistemic goals  $T_{d+1}$ , which makes a total of  $d\lceil \log_2 n \rceil + 1$  steps. ■

It is worth pointing out that determining whether a  $n$ -vertex graph  $G$  has the complete bipartite graph  $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$  as a subgraph can be achieved in polynomial time. To see this, firstly observe that any pair of vertices  $i, j$  of  $G$  which are not joined by an edge must be in the same part in the complete bipartite graph. In linear time, we can partition the vertices of  $G$  into subsets  $S_1, \dots, S_r$  such that vertices  $i, j$  not joined by an edge belong to the same set  $S_t$  (for some  $1 \leq t \leq r$ ). It only remains to test whether it is possible to partition the numbers  $|S_1|, \dots, |S_r|$  into two sets whose sums are  $\lceil n/2 \rceil$  and  $\lfloor n/2 \rfloor$ . This partition problem can be solved by dynamic programming in  $O(r(|S_1| + \dots + |S_r|))$  time and space, which is at worst quadratic since  $r \leq n$  and  $|S_1| + \dots + |S_r| = n$  [18]. On the other



**Figure 3.** The four steps in each pass of the parallel protocol for  $n = 13$ . The step on the right also occurs on its own as a preliminary step.

hand, it is known that deciding whether Directional-gossip(1) (the problem in which the digraph  $G$  is part of the input) can be solved in a given number of steps is NP-complete [19].

We now show that the solution plans given in the proof of Proposition 7 are optimal in the number of steps.

**Theorem 8** *The number of steps required to solve Parallel-gossip $_G(d)$  (for any graph  $G$ ) is at least  $d(\lceil \log_2 n \rceil - 1) + 1$  if  $n$  is even, or  $d\lceil \log_2 n \rceil + 1$  if  $n$  is odd.*

**Proof:** Consider any solution plan for Parallel-gossip $_G(d)$ . Recall that  $T_r$  is the conjunction of  $K_{i_1} \dots K_{i_r} s_{i_r}$  for all  $i_1, \dots, i_r \in \{1, \dots, n\}$ .

We give a proof by induction. For even  $n$ , suppose that at least  $r(\lceil \log_2 n \rceil - 1) + 1$  steps are required to establish  $T_{r+1}$ . This is true for  $r = 1$  because it takes at least a sequence of  $\lceil \log_2 n \rceil$  steps of calls for knowledge from any agent to reach  $n$  agents (thus establishing  $T_2$ ) [4, 20, 16]. For general  $r$  and without loss of generality, suppose that before the last step to establish it,  $T_{r+1}$  was false because of lack of knowledge of agent  $j$  (i.e.  $K_j T_r$  was false). By induction hypothesis this is at least the  $(r(\lceil \log_2 n \rceil - 1))$ -th step. A call in this step involves  $j$  and another agent, say  $i$ , and establishes not only  $T_{r+1}$ , but also  $K_j T_{r+1}$  and  $K_i T_{r+1}$ . However,  $\neg K_k T_{r+1}$  holds both before and after this step, for the agents  $k$  distinct from  $i$  and  $j$ . To establish  $T_{r+2}$ , it is necessary to distribute  $T_{r+1}$  from  $i$  and  $j$  to all other agents and this takes at least  $\lceil \log_2 n \rceil - 1$  steps (since each step can at most double the number  $m$  of agents having this knowledge and thus  $\lceil \log_2(n/2) \rceil$  steps are required to go from  $m = 2$  to  $m = n$ ). Hence, at least  $(r+1)(\lceil \log_2 n \rceil - 1) + 1$  steps are required to establish  $T_{r+2}$ . By induction on  $r$ , we obtain the lower bound  $d(\lceil \log_2 n \rceil - 1) + 1$ .

For odd  $n$ , the proof is similar but at least one more step is required for each epistemic level  $r$  because at least one agent doesn't communicate his knowledge on the first step to establish  $T_{r+1}$ . Hence, it takes at least a sequence of  $\lceil \log_2 n \rceil + 1$  steps for knowledge from all  $n$  agents to reach each others, and the lower bound is  $d\lceil \log_2 n \rceil + 1$ . ■

It is interesting to note that it can happen that increasing the number of secrets (and hence the number of agents) leads to less steps. Consider the concrete example of 7 or 8 agents. By Proposition 7 and Theorem 8, the number of steps decreases from  $3d+1$  to  $2d+1$  when the number of agents increases from 7 to 8. We can explain this by the fact that in the case of an odd number of agents, during each step there is necessarily one agent who is not communicating. By adding an extra agent, we can actually achieve a larger number of calls in a fewer number of steps.

## 6 Complexity of gossiping with negative goals

Not surprisingly, when we allow negative goals, the gossip problem becomes harder to solve. However, we will show that for several different versions of this problem, we avoid the PSPACE complexity of classical planning [6].

We also consider a slightly more general version of the gossip problem in which the maximum epistemic depth  $d$  is no longer a constant, but is part of the input. Let Gossip-pos and Gossip-neg be, respectively, the same as Gossip-pos( $d$ ) and Gossip-neg( $d$ ) in which there is no fixed bound  $d$  on the maximum epistemic depth of goal fluents. Although we do not specify the exact format in which the goals are given, we make the assumption that this requires at least  $n + d + m$  space, where  $m$  is the number of goal fluents. Recall that in these versions of the gossip problem, the graph  $G$  is also part of the input.

**Theorem 9** *Gossip-pos  $\in P$ . Indeed, if a solution plan exists, it can be found in polynomial time.*

**Proof:** The connected components of the graph  $G$  can be determined in polynomial time as can a spanning tree of each connected component. If there is a fluent  $K_{i_1} \dots K_{i_r} s_j$  in *Goal*, where the agents  $i_1, \dots, i_r, j$  do not all belong to the same connected component of  $G$ , then the planning problem has no solution. Otherwise, there is a solution obtained by applying the protocol given in the proof of Proposition 1 to each connected component and for a value of  $d$  equal to the maximum epistemic depth of goals. To construct this solution we only require knowledge of the spanning tree of each connected component. ■

When we allow negative goals the problem of deciding the existence of a solution plan becomes NP-complete.

**Theorem 10** *Gossip-neg and Gossip-neg(1) are both NP-complete.*

**Proof:** We first show that Gossip-neg  $\in$  NP. We will show that if a solution plan  $P$  exists then there is a solution plan  $P'$  of length no greater than  $md(n-1)$ , where  $m$  is the number of goal fluents and  $d$  the maximum epistemic depth of goal fluents. The validity of a plan of this length can clearly be verified in polynomial time.

Consider a goal  $g = K_{i_1} \dots K_{i_r} s_j$  (where  $r \leq d$ ). In  $P$  there must be a sequence of CALL $_{p,q}$  actions where the edges  $\{p, q\}$  in the graph  $G$  form a path from  $j$  to  $i_1$  passing through  $i_r, \dots, i_2$  in this order. There may be many such paths: for each goal  $g$  let  $path(g)$  be one such path. Divide  $path(g)$  into subpaths  $j \rightarrow i_r, i_r \rightarrow i_{r-1}, \dots, i_2 \rightarrow i_1$ . If any of these subpaths contains a cycle, this cycle can be

eliminated from  $path(g)$ . Call the resulting reduced path  $path'(g)$ . We can see that each subpath in  $path'(g)$  is of length no greater than  $n - 1$  (otherwise it would contain a cycle). Thus,  $|path'(g)| \leq r(n - 1) \leq d(n - 1)$ . Each goal  $g$  can therefore be achieved by a subset of the actions of  $P$  (corresponding to  $path'(g)$ ). Let  $P'$  be identical to  $P$  except that we only keep the actions  $CALL_{p,q}$  such that the corresponding edge  $\{p, q\}$  belongs to some  $path'(g)$ .  $P'$  then constitutes a valid plan and is of length at most  $md(n - 1)$ . It follows that Gossip-neg  $\in$  NP since the validity of a plan of this length can be verified in polynomial time. Trivially, we also have Gossip-neg(1)  $\in$  NP since Gossip-neg(1) is a subproblem of Gossip-neg.

To complete the proof, it suffices to give a polynomial reduction from the well-known NP-complete problem SAT to Gossip-neg(1). Let  $I_{SAT}$  be an instance of SAT. We will construct a graph  $G$  and a list of goals such that the corresponding instance  $I_{Gossip}$  of Gossip-neg(1) is equivalent to  $I_{SAT}$ . Recall that the nodes of  $G$  are the agents and the edges of  $G$  the communication links between agents.

For each propositional variable  $x$  in  $I_{SAT}$ , we add four nodes  $x, \bar{x}, b_x, d_x$  to  $G$  joined by the edges shown in Figure 4(b). There is a source node  $a$  in  $G$  and edges  $(a, x), (a, \bar{x})$  for each variable  $x$  in  $I_{SAT}$ . For each clause  $c_j$  in  $I_{SAT}$ , we add a node  $c_j$  joined to the nodes corresponding to the literals of  $c_j$ . This is illustrated in Figure 4(a) for the clause  $c_j = \bar{x} \vee y \vee z$ . The solution plan to  $I_{Gossip}$  will make the secret  $s_a$  transit through  $x$  (on its way from  $a$  to some clause node  $c_j$ ) if and only if  $x = \text{true}$  in the corresponding solution to  $I_{SAT}$ .

For each clause  $c_j$  in  $I_{SAT}$ ,  $G$  contains a clause gadget as illustrated in Figure 4(a) for the clause  $\bar{x} \vee y \vee z$ . We also add  $K_{c_j} s_a$  to the set of goals. Clearly, the secret  $s_a$  must transit through one of the nodes corresponding to the literals of  $c_j$  ( $\bar{x}, y$  or  $z$  in the example of Figure 4) to achieve the goal  $K_{c_j} s_a$ .

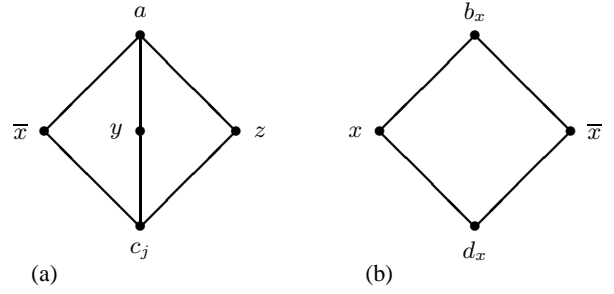
To complete the reduction, it only remains to impose the constraint that  $s_a$  transits through at most one of the nodes  $x, \bar{x}$ , for each variable  $x$  of  $I_{SAT}$ . This is achieved by the negation gadget shown in Figure 4(b) for each variable  $x$ . We add the goals  $K_{d_x} s_{b_x}, \neg(K_{d_x} s_a)$  for each variable  $x$ , and the goal  $\neg(K_{c_j} s_{b_x})$  for each variable  $x$  and each clause  $c_j$  (containing the literal  $x$  or  $\bar{x}$ ). The goal  $K_{d_x} s_{b_x}$  ensures that the secret  $s_{b_x}$  transits through  $x$  or  $\bar{x}$ . Suppose that  $s_{b_x}$  transits through  $x$ : then  $s_a$  cannot transit through  $x$  before  $s_{b_x}$  (because of the goal  $\neg(K_{d_x} s_a)$ ) and cannot transit through  $x$  after  $s_{b_x}$  (because of the goal  $\neg(K_{c_j} s_{b_x})$ ). By a similar argument, if  $s_{b_x}$  transits through  $\bar{x}$ , then  $s_a$  cannot transit through  $\bar{x}$ . Thus, this gadget imposes that  $s_a$  transits through exactly one of the nodes  $x, \bar{x}$ .

We have shown that  $I_{SAT}$  has a solution if and only if  $I_{Gossip}$  has a solution. Since the reduction is clearly polynomial, this completes the proof. ■

Our NP-completeness results are not affected by a restriction to one-way communication, i.e. Directional-gossip-neg and Directional-gossip-neg(1) are both NP-complete, by exactly the same proof as for Theorem 10. A similar remark holds for Parallel-gossip-neg and Parallel-gossip-neg(1).

## 7 Complexity of gossiping with variable secrets

Up to now we have assumed that the secrets  $s_i$  are constants. We now introduce a new kind of action  $CHANGE_i$  which simulates what happens when agent  $i$  changes his secret (which we imagine corresponds, for example, to his password). The effect of action  $CHANGE_i$  is to render all fluents of the form  $K_{i_1}, \dots, K_{i_r} s_i$  false,



**Figure 4.** (a) gadget imposing the clause  $c_j = \bar{x} \vee y \vee z$ ; (b) gadget imposing the negation  $\bar{x} = \neg x$ .

for  $i_r \neq i$ , since agent  $i_r$  does not know the new value of  $s_i$ . These new actions allow us to solve certain gossip problems which cannot be solved without them. For example, consider two agents and the set of goals  $\{K_{1s_2}, \neg K_{2s_1}\}$ . In Gossip-neg there is no solution to this planning problem, since the goal  $K_{1s_2}$  requires the action  $CALL_{1,2}$  which also establishes  $K_{2s_1}$ . However, the plan  $(CALL_{1,2}, CHANGE_1)$  achieves the goals  $K_{1s_2}$  and  $\neg K_{2s_1}$ . An example of this plan is exchanging telephone numbers with someone and then promptly changing one's own number. Denote by Gossip-neg-change the version of Gossip-neg with the new  $CHANGE_i$  actions. Although the  $CHANGE_i$  actions can help to solve more problems, it turns out that Gossip-neg-change is in the same complexity class as Gossip-neg, as we now prove.

**Theorem 11** *Gossip-neg-change is NP-complete.*

**Proof:** It is simple to verify that the reduction from SAT given in the proof of Theorem 10 remains valid: in the instances corresponding to instances of SAT, the actions  $CHANGE_a$  and  $CHANGE_{b_x}$  cannot be used without destroying goals which must be attained.

Thus, to complete the proof, it suffices to show that Gossip-neg-change  $\in$  NP. As in the proof of Theorem 10, it suffices to show that if a solution plan  $P$  exists, then there is a solution plan  $P'$  of length no greater than a polynomial function of  $m, d$  and  $n$ . To transform  $P$  into an equivalent plan  $P'$ , we can eliminate all useless actions. We consider an action  $a$  to be useless in  $P$  if all fluents  $K_{i_1} \dots K_{i_r} s_j$  it achieves were already true or  $CHANGE_j$  occurs after  $a$  in  $P$ . Since fluents  $K_{i_1} \dots K_{i_r} s_j$  can only become true at most once after the last occurrence of  $CHANGE_j$  in  $P$ , we can deduce that the number of actions in  $P'$  is bounded above by  $md(n - 1)$  (as in the proof of Theorem 10). If  $CHANGE_i$  occurs in  $P$ , then all its occurrences except the last can be deleted without affecting the validity of the plan. Thus the total number of actions in  $P'$  is bounded above by  $n + md(n - 1)$ , which completes the proof. ■

In the problem Gossip-neg-change, the  $CHANGE_i$  actions have no preconditions. If there are different actions  $CHANGE_i$  depending on the values of some subset of the secrets, then it is not difficult to see that we can simulate the version of classical STRIPS planning in which all actions have a single effect, which is known to be PSPACE-complete [6]. A more interesting avenue of future research is perhaps to investigate restricted versions of Gossip-neg or Gossip-neg-change which can be solved in polynomial time. As a simple example, suppose that the agents can be arranged in a hierarchy so that each agent  $i$  belongs to a level  $L_i$  and the goal is to communicate

STRIPS planning	PSPACE-complete
Gossip-neg-change	NP-complete
Gossip-neg	
Gossip-pos	polynomial

**Figure 5.** Complexity results for different decision versions of the gossip problem.

all secrets upwards in the hierarchy but not downwards. A solution consists in, for each level  $L$  in turn starting with the lowest level, all agents at this level communicate their secrets to all agents at level  $L + 1$  in the hierarchy, then all agents at level  $L + 1$  change their secrets so that the agents at level  $L$  no longer know these secrets. In this way all secrets percolate up the hierarchy but not down.

## 8 Discussion and conclusion

We summarize our complexity results in Figure 5. In each case the problem is the decision problem, i.e. testing the existence of a solution plan. The general conclusion that can be drawn from this figure is that many interesting epistemic planning problems are either solvable in polynomial time or are NP-complete, thus avoiding the PSPACE-complete complexity of planning. We consider the gossip problem to be a foundation on which to base the study of richer epistemic planning problems involving, for example, communication actions with preconditions involving the contents of the messages received by the agent. Previous work on temporal planning may help to provide a more realistic model of communication actions in which, for example, the length of a call is a function of the quantity of information exchanged, and correct communication during a telephone call requires concurrency of the speaking and listening actions of the two agents [9, 8].

Restricting our attention to the epistemic version of the classical gossip problem in which all positive epistemic goals of depth  $d$  must be attained, we have generalised many results from the classical gossip problem to the epistemic version. We have shown that for a complete graph  $G$ , no protocol exists which solves  $\text{Gossip}_G(d)$  in less than  $(d + 1)(n - 2)$  calls. This was known to be true for  $d = 1$  [3, 12]. We have given a protocol which uses only this number of calls (for any graph  $G$  containing  $K_{2,n-2}$  as a subgraph). In the case of one-way communications, we have again generalised the optimal protocol from the classical gossip problem to the epistemic version. This protocol requires only  $(d + 1)(n - 1)$  calls. When calls can be performed in parallel, and the aim is to minimise the number of steps rather than the number of calls, we have again generalised the optimal protocol from the classical gossip problem to the epistemic version. In this case, only  $O(d \log n)$  steps are required.

There remain many interesting open problems concerning the optimisation version of the gossip problem: given any graph  $G$ , determine the minimum number of calls required to attain a set of goals. For example, in the case of one-way communications, our optimal protocol requires a Hamiltonian path in a graph and detecting a Hamiltonian path is NP-complete [11]. However, it may be that

another optimal protocol exists which does not require the existence of a Hamiltonian path. A similar situation occurred in the case of two-way communications, in which we gave a protocol which depends on the existence of  $K_{2,n-2}$  as a subgraph, and this graph can be detected in polynomial time. The complexity of the problem of minimising the number of calls (whether two-way or one-way) in an arbitrary graph  $G$  is still open.

In this paper we have assumed a centralised approach in which a centralised planner decides the actions of all agents. Other workers have studied the classical gossip problem from a completely different perspective, assuming that all agents are autonomous [1, 25, 10]. An interesting avenue of future research would be to consider the generalised gossip problem in this framework.

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